Lecture 20

Monday, 14 November 2022 12:40 AM

Simple Near-Optimal Auctions.

Recall: by Revenue-Equivalence Principle, for DSIC mechanisms & monotone $(x_i(b_i))_{i\in N}$,

$$\sum_i \mathbb{E}[p_i(b_i)] = \sum_i \mathbb{E}[\varphi_i(b_i)\, x_i(b_i)]$$

hence, maximizing expected revenue is equivalent to maximizing expected virtual social welfare
If F_i's are regular, then can choose x "instance by instance": given b_i's, choose x to maximize $\sum_i \varphi_i(b_i) x_i$

This can however lead to counterintuitive mechanisms, eg. for 2 bidders if $v_1 \sim U[0,1]$, $v_2 \sim [0,2]$, the mechanism may give the single item to the lower bidder...

We want to come up with more "natural" auctions, but will pay the price in terms of revenue (only approximately optimal revenue)

Setting:
- n bidders, $v_i \sim F_i$ $\forall i$ independently
- single item to be auctioned (here $A = [n+1]$, since item may not be assigned to anyone)

want DSIC mechanisms, so x must be monotone, but may not maximize virtual social welfare.

Will first look at "prophet inequalities" & "stopping rules" for an online problem.

Online Bayesian Secretary Problem

- n known, bounded, continuous, regular distributions $G_1, \dots, G_n$
- at time step $i \in [n]$, sample $X_i \sim G_i$ is drawn & offered to us
- we can choose to accept & stop, or reject and look at the sample in the next time step
- if we reject all n samples, our value is zero.
- objective is to maximize expected value.

For any algorithm, we are interested in the competitive ratio:

$$\min_{G_1,\dots,G_n} \frac{\text{expected value for algorithm}}{\mathbb{E}_{X_i\sim G_i}[\max_i X_i]}$$

expected value obtained by adversary (called a "prophet") that knows all the random samples

Threshold Stopping Rules

Consider the following algorithm, parametrized by $t \in \mathbb{R}_+$:
- at time step i, if $X_i \geq t$, accept & stop, else reject.

Such an algorithm is known as a threshold stopping rule.

Theorem (Prophet Inequality): There exists $t \in \mathbb{R}_+$ such that the TSR w/ parameter t is at least ½-competitive, i.e., gets at least ½ the value obtained by the prophet (in expectation)

Proof: Given $t \in \mathbb{R}_+$, define $q(t) = \Pr[X_i < t \;\; \forall i] = \prod_{i=1}^{n} \Pr[X_i < t]$

Step 1: expected value of TSR

$$= \sum_i \underbrace{\Pr[X_i \geq t,\, X_j < t \;\; \forall j < i]}_{\text{Prob we stop at } i\text{th step}}\; \underbrace{\mathbb{E}[X_i \mid X_i \geq t,\, X_j < t \;\; \forall j<i]}_{\text{expected value, given that we stop at } i\text{th step}}$$

we will take out 't' from the expected value

$$= \sum_i \left(t + \mathbb{E}[X_i - t \mid X_i \geq t,\, X_j < t \;\; \forall j<i]\right) \Pr[X_i \geq t,\, X_j < t \;\; \forall j < i]$$

$$= t \underbrace{\sum_i \Pr[X_i \geq t,\, X_j < t \;\; \forall j<i]}_{(1-q(t))} + \sum_i \mathbb{E}[X_i - t \mid X_i \geq t,\, X_j < t \;\; \forall j<i] \Pr[X_i \geq t,\, X_j < t \;\; \forall j<i]$$

$$= t(1-q(t)) + \sum_i \mathbb{E}[X_i - t \mid X_i \geq t] \Pr[X_i \geq t] \left(\prod_{j<i} \Pr[X_j < t]\right)$$

$$\geq t(1-q(t)) + q(t) \sum_i \mathbb{E}[X_i - t \mid X_i \geq t] \Pr[X_i \geq t]$$

$$\geq t(1-q(t)) + q(t) \sum_i \mathbb{E}[\max\{X_i - t, 0\}] \quad \text{—— (I)}$$

Step 2: expected value of OPT:

$$= \mathbb{E}[\max_i X_i]$$

$$= t + \mathbb{E}[\max_i (X_i - t)]$$

$$\leq t + \mathbb{E}[\max_i \{\max\{X_i - t, 0\}\}]$$

$$\leq t + \sum_i \mathbb{E}[\max\{X_i - t, 0\}] \quad \text{—— (II)}$$

Now if we choose t s.t. $q(t) = ½$, we get that expected value of TSR $\geq \frac{1}{2}$ expected value of OPT ∎

We will use this to design a reserve price mechanism (for single-item setting) w/ expected revenue at least ½ the optimal revenue.

Recall that for a DSIC mechanism, by Revenue Equivalence,

$$\mathbb{E}_{v_i \sim F_i}\left[\sum_i p_i(v_i)\right] = \mathbb{E}_{v_i \sim F_i}\left[\sum_i \varphi_i(v_i)\, x_i(v_i)\right]$$

For regular distributions, the optimal revenue mechanism gives the item to bidder w/ highest non-zero virtual valuation.

For each bidder i, $v_i \sim F_i$, let ψ_i be the distribution of φ_i
Comparing w/ the Bayesian Secretary Problem,

$$\text{OPT} = \mathbb{E}_{\varphi_i \sim \psi_i}[\max_i \varphi_i]$$

Hence, by the Prophet Inequality theorem,
Choose t : $\prod_i \Pr[\varphi_i < t] = ½$
Let $t_i = \varphi_i^{-1}(t)$, i.e., $\varphi_i(t_i) = t$

Algo: order the bidders by decreasing bids.
give item to first bidder i s.t. $\varphi_i(b_i) \geq t$
(or $b_i \geq \varphi_i^{-1}(t) = t_i$)

(alternately, say bidder i "qualifies" if $\varphi_i(b_i) \geq t$
give item to qualified bidder w/ highest bid)

Recall that the optimal revenue auction may not even choose the highest bidder!
Here this auction is arguably more "natural" (though at the cost of revenue).

The Bulow-Klemperer Theorem (more simple auctions)

We started looking at mechanism design from the "prior-free" setting, where we don't know F_i's.

However for revenue maximization, in the single-bidder case, no DSIC mechanism can obtain non-trivial revenue.
Hence we switched to the Bayesian setting.

But is the single bidder case a solitary bad example?
Eg. if we have ≥ 2 bidders, the VSP auction is DSIC, gives non-trivial revenue.

So how bad is VSP, compared to the optimal revenue auction?

Setting:
- n bidders, $v_i \sim F$ iid (i.e., all bids drawn from the same distribution), F regular
- Single item to be auctioned
- let OPT_n be expected revenue of the optimal revenue auction w/ n bidders, VSP_n be expected revenue of the second price auction w/ n bidders.

Theorem (Bulow-Klemperer, 1996): $\text{VSP}_{n+1} \geq \text{OPT}_n$

Proof. Given the optimal auction for n agents, consider the following auction AUC_{n+1} for $n+1$ bidders:
- if OPT_n assigns item to bidder $i \in [n]$, so does AUC_{n+1}
- if OPT_n does not assign item to any bidder, AUC_{n+1} assigns item to bidder $n+1$.

Note that:
(I) revenue of AUC_{n+1} = revenue of OPT_n (verify!)
(II) AUC_{n+1} always assigns the item.

Now consider the optimal auction OPT^+_{n+1} that always assigns the item.
OPT^+_{n+1} assigns item to bidder w/ highest virtual valuation
$\Rightarrow$ it assigns item to highest bidder (since v_i's are iid, F regular)
Further, revenue of $\text{AUC}_{n+1} \leq$ revenue of OPT^+_{n+1}
Hence, revenue of $\text{OPT}^+_{n+1} \geq$ revenue of OPT_n
But note that OPT^+_{n+1} is just the VSP auction. ∎